ON THE MOTION OF A BODY WITH VARIABLE MASS, LIMITED POWER AND GIVEN TIME OF ACTION

(O DVIZHENII TELA PEREMENNOI MASSY S OGRANICHENNOI MOSHCHNOST'IU I ZADANNYM AKTIVNYM VREMENEM)

PMM Vol.27, No.5, 1963, pp.854-863

Iu.N. IVANOV
(Moscow)

(Received January 19, 1963)

Optimal motion regimes for bodies of variable mass with limited jet power were considered in [1-9]. There, the active time for jet operation was chosen optimal. The purpose of the present investigation is to generalize the previously obtained results for the case when the operating time of the powerplant is given and is less than optimal.

Section 1 discusses the general approach of solving the variational problem with a given time of control action less than optimal. Section 2 formulates the variational problem on the motion of a powerplant device with limited power, the working time of which (active time) is given and is less than optimal. Sections 3 and 4 illustrate the general results with analyses of optimal motions in a plane-parallel gravitational field. Two limiting cases of powerplant control are considered: an ideally controlled system (variable optimal thrust - Section 3), and an uncontrolled system (constant thrust - Section 4).

1. Let us consider the Mayer problem applicable to the dynamical system

$$\dot{x}_i = f_i(x_j, u_k) \qquad (i, j = 0, 1, \ldots, n;\ k = 1, \ldots, m) \qquad (1.1)$$

The quantities x_i, u_k are phase coordinates and control functions, respectively, differentiation being with respect to time t; the boundary conditions are defined at a given initial time ($t = 0$) and a final instant of time ($t = T$); the value of the phase coordinate $x_0(T)$ is the control functional subject to optimization. One of the control functions is bounded from below $u_1 \geqslant 0$. The control u_1 will be assumed switched on if $u_1 > 0$, and switched off if $u_1 = 0$; the sum of all intervals of time during which the control u_1 is switched on will be referred to as the control action time $T_м$.

Having solved the variational problem let us find the time $T_м^* \leqslant T$ which will be defined as the optimal action time for the control u_1.

Let, in addition to the above formulated variational problem, there be given a time of action for the control u_1 less than the optimal $T_м < T_м^*$.

In order to reduce the complicated variational formulation to the standard Mayer formulation, we will introduce an auxiliary phase coordinate $t_м$ which is the current time of action of the control u_1, and the control δ related by the differential equation $\dot{t}_м = \delta$. The control $\delta(t)$ is a relay function assuming a value of unity at the moment of switching on, and the value of zero when the control u_1 is switched off. Utilizing the properties of the function $\delta(t)$ we will replace the control u_1 by $u_1\delta$; this product coincides with u_1 during switching on and is zero when switched off. The system (1.1) becomes

$$\dot{x}_i = f_i(x_j, u_1\delta, u_k), \qquad \dot{t}_м = \delta \qquad (i, j = 0, 1, \ldots, n;\ k = 2, \ldots, m) \tag{1.2}$$

If, simultaneously with the above mentioned boundary conditions for the phase coordinates $x_i(0)$ and $x_i(T)$, the boundary conditions for the auxiliary coordinate $t_м$ are also satisfied

$$t_м(0) = 0, \qquad t_м(T) = \int_0^T \delta dt = T_м < T_м^*$$

while the relay control $\delta(t)$ along with the remaining controls is chosen optimal in the sense of the control functional $x_0(T)$, then the variational problem with the additional condition of given time $T_м < T_м^*$ will be solved. In other words, the optimal number of switching on operations and the optimal time of action for the u_1 control will be indicated in each active section of the trajectory. In solving the given problem by the L.S. Pontriagin method we construct, as usual, the Hamiltonian H and write the differential equations for the impulses [momenta] p_i

$$H = \sum_{i=0}^{n} p_i f_i(x_j, u_1\delta, u_k) + p_м\delta, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \qquad \dot{p}_м = 0 \tag{1.3}$$

Let us represent H as the function of the control δ as follows:

$$H = H_0 + (H_1 - H_0 + p_м)\,\delta$$

$$H_0 = \sum_{i=0}^{n} p_i f_i(x_j, 0, u_k), \qquad H_1 = \sum_{i=0}^{n} p_i f_i(x_j, u_1, u_k) \tag{1.4}$$

Let, for definiteness, there be a requirement to find the maximum of the control functional $x_0(T)$, i.e. the H function must be reaching the absolute minimum on the optimal controls u_1, u_k and δ. For the control δ the absolute minimum of H takes place when

$$\delta = 0 \quad \text{for } H_1 - H_0 + p_м > 0, \qquad \delta = 1 \quad \text{for } H_1 - H_0 + p_м < 0 \qquad (1.5)$$

The difference $H_1(t) - H_0(t)$ is nonpositive. Indeed, $H_1 = H_0$ for $u_1 = 0$, as follows from the definition (1.4) of the functions H_0, H_1; for other values of u_1 the difference $H_1 - H_0$ should be negative since, otherwise, the Hamiltonian H can be decreased by letting $u_1 = 0$, i.e. $H_1 - H_0 = 0$. This determines the sign for the impulse $p_м$

$$p_м > 0 \qquad (1.6)$$

($p_м < 0$ for the case of maximum H). If $p_м < 0$, then the expression $H_1 - H_0 + p_м$ would never change sign, and $\delta(t) \equiv 1$; at the same time $\dot{t}_м \equiv 1$ and $t_м(T) = T$, which would automatically violate the boundary condition $t_м(T)_* = T_м$. We note that for $u_1 = 0$ the optimal value for $\delta = 0$, since $p_м > 0$. If $p_м = 0$ then the resulting time $T_м = T_м{}^*$.

The described approach is also applicable for several controls with given times of action less than optimal. In that case, a required number of auxiliary controls δ is added instead of one.

2. 1. Let us introduce the following notation: G_m, G_n, G_Σ, G_N and G represent the current weight of the working medium, the payload, the sum of these two weight components, the weight of the powerplant, and the current weight of the body of variable mass, respectively; q, V, P and N are the weight consumption of the working medium, flow velocity, thrust, and the power of the jet, respectively; N_0 and α are the maximum power delivered by the powerplant and the unit weight of the powerplant; and a is the acceleration resulting from the jet.

The above quantities are related as follows:

$$G = G_\Sigma + G_N, \qquad G_\Sigma = G_m + G_n, \qquad N = \frac{qV^2}{2g}, \qquad P = \frac{qV}{g}$$

$$G_N = \alpha N_0, \qquad a = \frac{Pg}{G} = \frac{\sqrt{2gNq}}{G_\Sigma + G_N}$$

In the following we will use the weight characteristics referred to the initial weight of the body of variable mass with previous notations G_m, G_n, G_Σ, G_N, G and q, where the initial weight is unity. The power will be referred to the maximal power, retaining the notation N. Then the expression for acceleration will be

$$a = a(N,\ G_N,\ G_\Sigma,\ q) = \sqrt{(2g/\alpha) N G_N q} \,/\, (G_\Sigma + G_N) \tag{2.1}$$

Let us now consider the rectangular coordinates x, y and the corresponding velocities along the axes $\dot{x} = v_x$, $\dot{y} = v_y$, and denote by $X(x, y, t)$, $Y(x, y, t)$ the projections of the gravitational acceleration on the axes of the rectangular system of coordinates. The thrust direction will be characterized by the angle β between the thrust vector and the x-axis. The upper indexes 0 and 1 will refer to the beginning ($t = 0$) and end ($t = T$) of the motion, respectively.

The equations of plane motion of a body of variable mass in an arbitrary gravitational field and the boundary conditions are of the form

$$\dot{G}_\Sigma = -q, \quad \dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{v}_x = a\cos\beta + X, \quad \dot{v}_y = a\sin\beta + Y$$
$$G_\Sigma(0) = 1 - G_N{}^0, \quad x(0) = x^0,\ y(0) = y^0, \quad v_x(0) = v_x{}^0, \quad v_y(0) = v_y{}^0$$
$$x(T) = x^1, \quad y(T) = y^1, \quad v_x(T) = v_x{}^1, \quad v_y(T) = v_y{}^1 \tag{2.2}$$

where the function $a = a(N, G_N, G_\Sigma, q)$ is given by the formula (2.1).

In the considered problem the functions $\beta(t)$, $q(t)$, $N(t)$ and $G_N(t)$ are the controlling functions. In regard to G_N it is known [7] that the optimal programming of it along the trajectory is insignificant for the result. Therefore, in the following we will let G_N = const and will determine its value from the optimal conditions. The control $N(t)$ is bounded from below and above $0 \leqslant N(t) \leqslant 1$. The weight consumption $q(t)$ can be programmed either along the trajectory, if there is no restriction on the thrust, or assumed constant if the thrust P and the power N are constant.* Also, the consumption control may consist of sections where $q = 0$. The control $\beta(t)$ is not restricted in any way.

Let the dynamic system be subject to the equations and boundary conditions (2.2), and let there be given a time T and the active time $T_м$. Also, let the controls $\beta(t)$, $q(t)$, $N(t)$ be chosen from a permissible class. It is required to find optimal controls and optimal trajectories yielding a maximum of the functional $G_\Sigma{}^1 = G_n$ which is the relative payload.

Let us introduce the auxiliary phase coordinate $t_м$ and the control function δ and form the control $q\delta$ instead of the previous q. The complete system of equations and boundary conditions for $t_м$ are in this

---

* It will be shown below that the maximal utilization of power, i.e. $N = 1$, is optimal.

case of the form

$$\dot{G}_\Sigma = -q\delta, \qquad \dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = a\delta\cos\beta + X$$
$$\dot{v}_y = a\delta\sin\beta + Y, \qquad \dot{t}_м = \delta \qquad (t_м(0) = 0,\ t_м(T) = T_м) \tag{2.3}$$

The Hamiltonian H is in explicit form and the differential equations for the impulses p are expressed as follows:

$$H = -p_\Sigma q\delta + p_x v_x + p_y v_y + p_{vx}\left(\frac{\sqrt{(2g/\alpha)G_N qN}}{G_\Sigma + G_N}\delta\cos\beta + X\right) + p_{vy}\left(\frac{\sqrt{(2g/\alpha)G_N qN}}{G_\Sigma + G_N}\delta\sin\beta + Y\right) + p_м\delta \tag{2.4}$$

$$\dot{p}_\Sigma = (p_{vx}\cos\beta + p_{vy}\sin\beta)\frac{\sqrt{(2g/\alpha)G_N qN}}{(G_\Sigma + G_N)^2}\delta, \qquad \dot{p}_x = -p_{vx}\frac{\partial X}{\partial x} - p_{vy}\frac{\partial Y}{\partial x}$$
$$\dot{p}_y = -p_{vx}\frac{\partial X}{\partial y} - p_{vy}\frac{\partial Y}{\partial y}, \qquad \dot{p}_{vx} = -p_x, \qquad \dot{p}_{vy} = -p_y, \qquad \dot{p}_м = 0 \tag{2.5}$$

The final value of the impulse $p_\Sigma^1 = -1$. In the variational problem, one looks for the maximum of the final quantity G_Σ^1. Therefore, the sought optimal controls must yield a minimum of the Hamiltonian H.

The optimal controls $\beta(t)$ and $N(t)$ were given in [2,5,8-11]; in the present notation they are of the form

$$p_{vx} = -p_v\cos\beta, \qquad p_{vy} = -p_v\sin\beta \qquad \left(p_v = \sqrt{p_{vx}^2 + p_{vy}^2}\right) \tag{2.6}$$
$$N(t) \equiv 1 \tag{2.7}$$

Utilizing (2.6) and (2.7), we rewrite the equation for p_Σ as well as the function H, retaining in the latter the terms with the control functions

$$\dot{p}_\Sigma = -p_v\frac{\sqrt{(2g/\alpha)G_N q}}{(G_\Sigma + G_N)^2}\delta, \qquad H^* = \left(-p_\Sigma q - p_v\frac{\sqrt{(2g/\alpha)G_N q}}{G_\Sigma + G_N} + p_м\right)\delta \tag{2.8}$$

2. Let us consider the case of variable optimal consumption (thrust). If no restrictions of some kind are placed upon the consumption control then, as is known [2,4,7], the solution of the formulated variational problem is reduced to determination of the optimal law for variation of the thrust acceleration vector which results in the minimum of the functional

$$J = \int_0^T a^2 dt \tag{2.9}$$

while the optimal quantity G_N and the maximal quantity G_n are found from the relationships

$$G_N = \sqrt{\alpha J / 2g} - \alpha J / 2g, \qquad G_n = (1 - \sqrt{\alpha J / 2g})^2 \tag{2.10}$$

according to the known functional J.

Thus, the original problem of finding the maximum $G_\Sigma{}^1$ can be replaced in the case of the variable optimal consumption by the problem of finding the minimum J, and the control function $q(t)$ by the control function $a(t)$.

The problem of the given time of action $T_м < T_м{}^*$ in terms of the functional J and the control function $a(t)$ is described by the following system of differential equations:

$$\begin{gathered} \dot{J} = a^2\delta, \qquad \dot{x} = v_x, \qquad \dot{y} = v_y \\ \dot{v}_x = a\delta\cos\beta + X, \qquad \dot{v}_y = a\delta\sin\beta + Y, \qquad \dot{t}_м = \delta \end{gathered} \tag{2.11}$$

The Hamiltonian H and the differential equations for the impulses are of the form

$$H = -a^2\delta + p_x v_x + p_y v_y + p_{vx}(a\delta\cos\beta + X) + p_{vy}(a\delta\sin\beta + Y) + p_м\delta \tag{2.12}$$

$$\begin{gathered} \dot{p}_x = -p_{vx}\frac{\partial X}{\partial x} - p_{vy}\frac{\partial Y}{\partial x}, \qquad \dot{p}_y = -p_{vx}\frac{\partial X}{\partial y} - p_{vy}\frac{\partial Y}{\partial y} \\ \dot{p}_{vx} = -p_x, \qquad \dot{p}_{vy} = -p_y, \qquad \dot{p}_м = 0 \end{gathered} \tag{2.13}$$

The controls $a(t)$ and $\beta(t)$, which on the active sections yield a maximum of the function H, satisfy the relationships

$$a = p_v/2; \qquad p_{vx} = p_v\cos\beta, \qquad p_{vy} = p_v\sin\beta \qquad (p_v = \sqrt{p_{vx}^2 + p_{vy}^2}) \tag{2.14}$$

The times of switching on the acceleration are related to the change in the sign of the combination Δ

$$\delta = 1 \quad \text{for } \Delta > 0, \qquad \delta = 0 \quad \text{for } \Delta < 0 \qquad (\Delta = p_v{}^2/4 + p_м) \tag{2.15}$$

The quantity $p_м < 0$ determines the value of the control time of action $T_м$. If $T_м$ is not given beforehand, then $p_м = 0$ and $\Delta < 0$ and, consequently, there are no passive sections on the trajectory [8]. This conclusion is valid only in connection with the case of variable optimal consumption.

3. Let us consider the case of constant thrust. The optimal nature of the limiting control $N(t) \equiv 1$ along the active sections of the

trajectory was shown above. This property, along with the requirement of constant thrust, leads to the constancy of consumption $q(t)$ = const. Let us express for the case of constant thrust the equations of motion and the functional by means of a new control parameter, the initial acceleration a_0 due to the thrust. Since q = const, the consumption equation (2.3) is integrated and the relative payload is expressed as follows:

$$G_n = 1 - G_N - qT_м = 1 - (\varkappa / 2g)\, a_0^2 / q - qT_м \qquad (2.16)$$

System (2.3), without the first equation and with the aid of the parameter a_0, can be expressed as

$$\dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{v}_x = \frac{a_0\delta}{1 - qt_м}\cos\beta + X, \quad \dot{v}_y = \frac{a_0\delta}{1 - qt_м}\sin\beta + Y, \quad \dot{t}_м = \delta \qquad (2.17)$$

The impulse equations remain as before (see (2.5)) with the exception of the equation for p_Σ (2.8)

$$\dot{p}_\Sigma = -p_v \frac{a_0\delta}{(1 - qt_м)^2} \qquad (2.18)$$

The function H^* becomes

$$H^* = \left(-p_\Sigma q + p_v \frac{a_0}{1 - qt_м} + p_м\right)\delta \qquad (2.19)$$

The optimal control β is found with the aid of the impulses p_{vx}, p_{vy} (2.6). The instants of switching on ($\delta = 1$) and switching off ($\delta = 0$) coincide with the instants of sign change in the expression Δ

$$\delta = 1 \quad \text{for } \Delta < 0, \qquad \delta = 0 \quad \text{for } \Delta > 0 \qquad \left(\Delta = -p_\Sigma q - p_v \frac{a_0}{1 - qt_м} + p_м\right) \qquad (2.20)$$

The procedure of choosing the optimal value for G_N or, equivalently, the establishment of an optimal relation between a_0 and q, in contrast to the example worked out in Section 2, requires the simultaneous solution of the weight and trajectory parts of the general problem. The particular examples of such a procedure are given in Section 4.

In contrast to the case of variable optimal thrust, in this case it is not possible to prove the optimal trajectories without passive sections. It will be shown below (Section 4) that the inclusion of a passive section improves the result.*

* For motion in a force-free field, this conclusion was made by Preston-Thomas (see, for example [1]). Paper [6] gives a choice of an optimal passive section in the computation of interplanetary trajectories.

The number of passive sections on the trajectory is determined by the multivalence of the inverse function $p_v(t)$. Indeed, the first term $p_\Sigma q$ is unchanged in the expression for Δ on the passive section, but only the function $p_v(t)$ is changing and the number of roots of the equation $\Delta = 0$ depends on the form of this function.

3. Let us consider a motion in a plane-parallel gravitational field with variable optimal thrust. In the previously established notation, the motion is described by a system of differential equations (2.11) for $X(x, y, t) = 0$, $Y(x, y, t) = -g = \text{const}$. The differential equations for the impulses (2.13) can be integrated

$$p_x = c_1, \quad p_y = c_2, \quad p_{vx} = -c_1 t + c_3, \quad p_{vy} = -c_2 t + c_4 \tag{3.1}$$

From the expression for p_{vx} and p_{vy} and, consequently, for $p_v = [(-c_1 t + c_3)^2 + (-c_2 t + c_4)^2]^{1/2}$ it may be concluded that the trajectory in a plane-parallel field with a switch off control contains one passive section. Indeed, equation $\Delta = 0$ (see (2.15)) is quadratic and has two roots

$$t_{\kappa,\,н} = \frac{1}{c_1^2 + c_2^2}\left[c_1 c_3 + c_2 c_4 \pm \sqrt{-4 p_м (c_1^2 + c_2^2) - (c_1 c_4 + c_2 c_3)^2}\right] \tag{3.2}$$

which determine the beginning of $t_н$ and the end t_k of the single passive section. Since the duration of the latter is given as $t_k - t_н = T - T_м$, then the impulse $p_м$ can be eliminated from formulas (3.2) and the instants of beginning and end of the passive section are finally given by

$$t_{\kappa,\,н} = \frac{c_1 c_3 + c_2 c_4}{c_1^2 + c_2^2} \pm \frac{1}{2}(T - T_м) \tag{3.3}$$

Optimal laws for projections of the motor acceleration $a_x = a \cos\beta$, $a_y = a \sin\beta$ on the active sections are linear in time (see (2.14) and (3.1))

$$a_x = \tfrac{1}{2}(-c_1 t + c_3), \qquad a_y = \tfrac{1}{2}(-c_2 t + c_4) \tag{3.4}$$

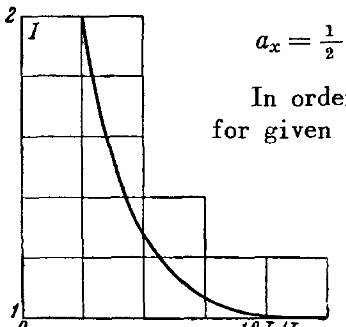

Fig. 1.

In order to find the constants c_1, c_2, c_3 and c_4 for given initial and boundary values of the coordinates and velocities, the equation of motion in system (2.11) should be integrated along the active and passive sections, the location of which is determined by formulas (3.3). The change in the projections of the motor acceleration is given by the functional dependence (3.4) on the active part of the trajectory, and at the instant of switching off the control, the motion becomes unaccelerated with $a_x = a_y = 0$.

As an example illustrating the method for solving the problem of the powerplant with switch-off capability, let us consider the one-dimensional motion in a force-free field ($g = 0$) between two positions of rest separated by a distance l. The boundary values of the phase coordinates are as follows:

$$x(0) = v_x(0) = v_x(T) = 0, \qquad x(T) = l \tag{3.5}$$

The beginning and end of the trajectories are the points of equilibrium; therefore, the passive section cannot start or end the motion. After carrying out a consecutive integration of the equations of motion (2.11) along the active and passive sections we obtain the constants c_1 and c_3

$$c_1 = 24\frac{l}{T^3 - (T - T_м)^3}, \qquad c_3 = 12\frac{Tl}{T^3 - (T - T_м)^3} \tag{3.6}$$

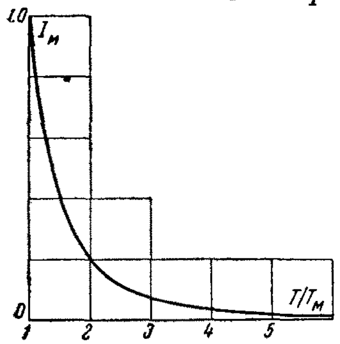

Fig. 2.

Substituting these quantities into (3.3) we find the time of start and end of the passive section

$$t_н = \tfrac{1}{2} T_м, \quad t_к = T - \tfrac{1}{2} T_м$$

as well as the projections of the motor accelerations

$$a_x = 6\frac{l(-2t + T)}{T^3 - (T - T_м)^3} \quad \text{for} \quad \begin{cases} T - \tfrac{1}{2}T_м \geqslant t \geqslant 0 \\ T \geqslant t \geqslant T - \tfrac{1}{2}T_м \end{cases}$$

$$a_x = 0 \quad \text{for} \quad T - \tfrac{1}{2}T_м \geqslant t \geqslant \tfrac{1}{2}T_м \tag{3.7}$$

The function $a_x(t)$ determines the value of the integral functional (2.9)

$$I = \int_0^{\frac{1}{2}T_м} a_x^2\,dt + \int_{T-\frac{1}{2}T_м}^{T} a_x^2\,dt = 12\frac{l^2}{T^3 - (T - T_м)^3} \tag{3.8}$$

Thus, the passive section on a trajectory connecting two points of rest in a force-free field is located in the middle of the trajectory. For a given time of motion T and given distance l between the points, the J integral increases monotonically for decreasing time $T_м$ of control action (see Fig. 1, $I = J/(12l^2/T^3)$). For a given time $T_м$ and given l, the J integral decreases monotonically for increasing time T (see Fig. 2, $I_м = J/(12l^2/T_м^3)$).

As a second example, let us consider the problem of acquiring the given absolute value of velocity for a body of variable mass in a force-free field. Let the body of variable mass begin its motion from an initial position with the fixed phase coordinates

$$x(0) = x^0, \quad y(0) = y^0, \quad v_x(0) = v_x^0, \quad v_y(0) = v_y^0$$

It is required to determine the optimal law for variation of the functions $a_x(t)$, $a_y(t)$ and $\delta(t)$ for given time of motion T, time of control action $T_м$ and the magnitude of the final velocity $v^1 = \sqrt{[(v_x^1)^2 + (v_y^1)^2]}$. The coordinates of end motion and the direction of the final velocity vector are not fixed but are chosen from the optimal conditions Therefore, $p_x(T) = p_y(T) = 0$, and in the formulas (3.1) $c_1 = c_3 = 0$.

Consequently, the function Δ (see (2.15) and (3.2)), which determines the times of switching on and off the control, is constant. However, the sign and magnitude of this function are not determined. This means that the distribution of passive sections of the trajectory is not determined and does not affect the functional of the problem. The projections of the motor acceleration are constant [2]

$$a_x = {}^1/_2\, c_3, \qquad a_y = {}^1/_2\, c_4 \tag{3.9}$$

The values of the parameters c_2 and c_4 and the expression for the functional are as follows:

$$c_3 = 2v_x^0 w \,/\, T_м, \qquad c_4 = 2v_y^0 w \,/\, T_м, \quad J = w^2 \,/\, T_м \qquad (w = v^1 \,/\, v^0 - 1) \tag{3.10}$$

4. Let us consider the motion in a plane-parallel field with constant thrust. In this case, the equations of motion (2.17) and the impulse equations are simplified. The impulses p_{vx} and p_{vy} are expressed analogously to the case of variable optimal thrust (3.1). These expressions yield the formulas for the direction cosines of the thrust vector on the active sections of the trajectory

$$\sin\beta = \frac{c_2 t - c_4}{\sqrt{(-c_1 t + c_3)^2 + (-c_2 t + c_4)^2}}, \quad \cos\beta = \frac{c_1 t - c_3}{\sqrt{(-c_1 t + c_3)^2 + (-c_2 t + c_4)^2}} \tag{4.1}$$

The instants for switching on and off the thrust can be found from the conditions (2.20). Since the function $t(p_v)$ is in general a double valued function of its argument for a plane-parallel field, the trajectory can have no more than one passive section.

Indeed, if the combination Δ (2.20) at time $t_н$ changes sign from a minus to a plus, then starting with that time the motion becomes unaccelerated and only the function $p_v(t)$ can vary with time in the expression for $\Delta(t)$. For a double valued $t(p_v)$ there will exist a time t_k such that $p_v(t_k) = p_v(t_н)$ and then the combination Δ will change sign for the second time. The beginning $t_н$ and the end t_k for the passive section are found according to formulas similar to (3.2). It is worth noting that the conclusion about the number of passive sections in a plane-parallel field coincides with the above obtained result in Section 3 for the variable optimal thrust as well as for constant thrust but for an entirely different class of motive systems [10,11].

Let us consider both problems given in Section 3 under the condition of constant thrust. The first problem is one-dimensional motion between two points of rest separated by a distance l from each other. The one-dimensional motion takes place for the values of the constants $c_2 = c_4 = 0$ in formulas (4.1). Also $\sin\beta = 0$, $\cos\beta = \pm 1$, i.e. the thrust vector has a direction which coincides with the direction of motion or is opposite to it.

In order to establish the number and instant of thrust direction changes, we will again formulate a variational problem for a one-dimensional motion by introducing the control $\beta^*(t) = \pm 1$ for the direction of the thrust vector. The equations of motion, the equations for the impulses, and the Hamiltonian are in this case, expressible as (see (2.17), (2.28) and (2.19))

$$\dot{G}_\Sigma = -q\delta, \qquad \dot{x} = v_x, \qquad \dot{v}_x = \frac{a_0\delta}{1 - qt_м}\beta^*, \qquad \dot{t}_м = \delta$$

$$\dot{p}_\Sigma = -\dot{p}_{vx}\frac{a_0\delta}{(1 - qt_м)^2}\beta^*, \qquad \dot{p}_x = 0, \qquad \dot{p}_{vx} = -p_x, \qquad \dot{p}_м = 0 \tag{4.2}$$

$$H = -p_\Sigma q\delta + p_x v_x + p_{vx}\frac{a_0\delta}{1 - qt_м}\beta^* + p_м\delta$$

For a minimum of the function H, it is required that $\beta^*(t) = -\operatorname{sign} p_{vx}$. Inasmuch as the impulse p_{vx} is a linear function of time

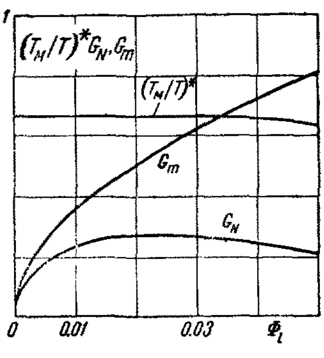

Fig. 3.

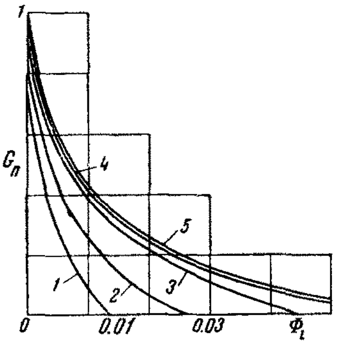

Fig. 4.

$p_{vx} = -c_1 t + c_3 = c(t - t^*)$, then the thrust vector changes direction in a single-valued manner at the instant $t = t^*$. The combination Δ which determines the boundaries of the passive section is, for the case considered, of the form

$$\Delta = -p_\Sigma q - |c(t - t^*)| \frac{a_0}{1 - qt_м} + p_м \tag{4.3}$$

It follows from the analysis of the roots of the equation $\Delta = 0$ which determines the beginning and end of the passive section that $t^* - t_н = t_k - t^*$, i.e. the time t^* divides the passive section into two equal parts. Thus, the optimal trajectory of translation from one point of rest to another with constant thrust, as in the previous case, consists of three sections: the acceleration section ($0 \leqslant t \leqslant t_н$, $\delta = 1$, $\beta^* = 1$), the passive section ($t_н \leqslant t \leqslant t_k$, $\delta = 0$), and the deceleration section ($t_k \leqslant t \leqslant T$, $\delta = 1$, $\beta^* = -1$).

The integration of the equations of motion sequentially along the three sections yeidls two relations between the beginning $t_н$ and the duration $T_м$ of the active sections in terms of the consumption q and the initial motor acceleration a_0

$$1 - qT_м = (1 - qt_н)^2, \qquad l = \frac{a_0}{q}\left[2t_н - T_м - \frac{1}{2}(T - T_м)\ln(1 - qT_м)\right] \tag{4.4}$$

In deriving the second formula in (4.4), the quantity t_k was eliminated with the aid of the relationship $t_k = t_н + T - T_м$. Expressing $t_н$ in the first formula of (4.4) by q, $T_м$ and substituting into the second one, we utilize the obtained relationship for elimination of a_0 from the expression for the functional G_n (2.16)

$$G_n = 1 - G_m - \Phi_l (T / T_м)^3 G_m^3 \quad [2(1 - \sqrt{1 - G_m}) - G_m +$$
$$+ {}^1/_2 G_m (T / T_м - 1) \ln(1 - G_m)]^{-2} \tag{4.5}$$

where $G_m = qT_м$ is the supply of the working medium

$$\Phi_l = (\alpha / 2g)\, l^2 / T^3$$

The procedure for finding the maximum G_n has been carried out for several given values of $T_м/T$, as well as for the optimal $(T_м/T)^*$. The optimal duration of the active sections $(T_м/T)^*$ and the optimal relationship between the relative weights of the powerplant G_N and the supply of the working medium G_m are shown in Fig. 3 as function of the quantity Φ_l.

Figure 4 shows comparative curves $G_n(\Phi_l)$ for the following cases: (1) motion with constant thrust $T_м/T = 0.1$; (2) motion with constant thrust $T_м/T = 0.2$; (3) motion with constant thrust and optimal active time (the curve corresponds to the results of [1]); (4) motion with variable optimal thrust whose active time is equal for each value of Φ_l to the optimal active time for the case of constant thrust; (5) motion with variable optimal thrust without passive sections (Curve 5 corresponds to the results of [4,5]).

Curves 4 and 5 in Fig. 4 were computed from the expressions (see (2.10) and (3.8))

$$G_n = (1 - \sqrt{12\Phi_l / [1 - (1 - T_м / T)]^3})^2, \qquad G_n = (1 - \sqrt{12\Phi_l})^2 \qquad (4.6)$$

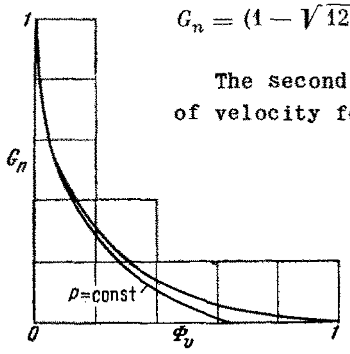

Fig. 5.

The second problem is the attaining of a given modulus of velocity for the motion in a force-free field [1]. The analysis of this problem is in many respects like that in Section 3. The values of the constants c_1 and c_2 were chosen for optimal reasons. The zero values were chosen likewise.

The combination Δ (2.20) which determines the times of switching off and on is a function independent of time along the passive section (since p_v = const). Therefore, the passive section can only close the trajectory and, consequently, only the active time $T_м$ affects the functional of the problem.

The results are shown in Fig. 5 where the relationship $G_n(\Phi_v)$ is given. The quantity Φ_v is expressed as

$$\Phi_v = (\alpha / 2g)(v^1 - v^0)^2 / T_м$$

The same figure shows, for comparison, the curve for the case of optimally variable consumption $G_n = (1 - \sqrt{\Phi_v})^2$ (see (2.10 and (3.10)).

BIBLIOGRAPHY

1. Preston-Thomas, H., Some design parameters of a simplified ion rocket. *J. Brit. Interplan. Soc.*, Vol. 116, No. 10, 1958.

2. Irving, J.H. and Blum, E.K., Comparative performance of ballistic and low-thrust vehicles for flight to Mars. *Vistas in Astronautics, 2, Second Annual Astronautics Symposium*, 1959.

3. Irving, I.H., Low-thrust flight: variable exhaust velocity in gravitational fields. *Space Technology*, 1959.

4. Grodzovskii, G.L., Ivanov, Iu.N. and Tokarev, V.V., O dvizhenii tela peremennoi massy s postoiannoi zatratoi moshchnosti v gravitatsionnom pole (On the motion of a body of variable mass with

constant use of power in a gravitational field). *Dokl. Akad. Nauk SSSR*, Vol. 137, No. 5, 1961.

5. Leitmann, G., Minimum transfer time for a power-limited rocket. *J. Appl. Mech.*, Vol. 28, Ser. E, No. 2, 1961.

6. Mackay, T.A., Rossa, L.G. and Zimmerman, A.V., Optimum low-acceleration trajectories for Earth-Mars transfer. Presented at IAS meeting on *Vehicle Systems Optimization*, Garden City, November 28-29, 1961.

7. Ivanov, Iu.N., Optimal'noe izmenenie moshchnosti pri dvizhenii tela peremennoi massy v gravitatsionnom pole (Optimal variation of power during the motion of a body of variable mass in a gravitational field). *PMM* Vol. 26, No. 4, 1962.

8. Isaev, V.K. and Sonin, V.V., Ob odnoi nelineinoi zadache optimal'nogo upravleniia (On a nonlinear problem of optimal control). *Avtomatika i Telemekhanika*, Vol. 23, Nos. 9-10, 1962.

9. Melbourn, W.G. and Sauer, C.G., Optimum thrust programs for power-limited propulsion systems. *Astronaut. Acta*, Vol. 8, Fasc. 4, 1962.

10. Isaev, V.K., Printsip maksimuma L.S. Pontriagina i optimal'noe programmirovanie tiagi raket (The L.S. Pontriagin maximum principle and optimal programming of rocket thrust). *Avtomatika i Telemekhanika*, Vol. 25, No. 8, 1961.

11. Isaev, V.K., Dopolnenie k rabote "Printsip maksimuma L.S. Pontriagina i optimal'noe programmirovanie tiagi raket" (Addendum to the paper "The L.S. Pontriagin maximum principle and optimal programming of rocket thrust"). *Avtomatika i Telemekhanika*, Vol.25, No. 1, 1962.

Translated by V.C.